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On the m-Fold Section of the Cyclotomic Equation in the Case of m Prime.

By J. C. Glashan, *Ottawa, Canada.*

Let m be any odd prime number, p a prime $= mn + 1$, t a primitive root of p,

$$(\chi^p - 1)/(\chi - 1) = 0 \tag{i}$$

and

$$\eta_h = \chi^{t^h} + \chi^{t^{h+m}} + \chi^{t^{h+2m}} + \dots + \chi^{t^{h+(n-1)m}}, \tag{ii}$$

$$h = 0, 1, 2, \dots, m-1$$

then will

$$1 + \eta_0 + \eta_1 + \eta_2 + \eta_3 + \dots + \eta_{m-1} = 0. \tag{iii}$$

Let

$$\eta_0\eta_r = e_{r,\,0}\eta_0 + e_{r,\,1}\eta_1 + e_{r,\,2}\eta_2 + \dots + e_{r,\,c}\eta_c + \dots e_{r,\,m-1}\eta_{m-1} \tag{iv}$$

$$r = 1, 2, 3, \dots, m-1.$$

On making $\chi = 1$ in (ii) and (iv), which may be done, r being > 0 and therefore (iii) not occurring in the reduction of $\eta_0\eta_r$ to the form $e_{r,\,0}\eta_0 + e_{r,\,1}\eta_1 + \dots$, we obtain

$$e_{r,\,0} + e_{r,\,1} + e_{r,\,2} + \dots + e_{r,\,m-1} = n. \tag{v}$$

Writing $m - r$ for r in (iv), it becomes

$$\begin{aligned}\eta_0\eta_{m-r} = {} & e_{m-r,\,0}\eta_0 + e_{m-r,\,1}\eta_1 + \dots + e_{m-r,\,c-r}\eta_{c-r} \\ & + \dots + e_{m-r,\,m+g-r}\eta_{m+g-r} + \dots + e_{m-r,\,m-1}\eta_{m-1}\end{aligned} \tag{vi}$$

$$c \not< r, \quad g < r$$

But p being prime, $\chi^t, \chi^{t^2}, \dots, \chi^{t^{p-2}}$ are prime roots of $\chi^p - 1 = 0$, and therefore the equations of form (iv) are all subject to the substitutions $(\eta_0\eta_1\eta_2 \dots \eta_{m-1})^h$. Operating on (iv) with $(\eta_0\eta_1\eta_2 \dots \eta_{m-1})^{m-r}$, it becomes

$$\begin{aligned}\eta_0\eta_{m-r} = {} & e_{r,\,r}\eta_0 + e_{r,\,r+1}\eta_1 + e_{r,\,r+2}\eta_2 + \dots + e_{r,\,c}\eta_{c-r} + \dots \\ & + e_{r,\,g}\eta_{m+g-r} + \dots + e_{r,\,r-1}\eta_{m-1}.\end{aligned} \tag{vii}$$

Comparing equations (vi) and (vii) we obtain

$$\left.\begin{aligned} e_{m-r,\, m+c-r} &= e_{r,\, c} \text{ if } c < r, \\ e_{m-r,\, c-r} &= e_{r,\, c} \text{ if } c = \text{ or } > r. \end{aligned}\right\} \tag{viii}$$

From equations (iii) and (iv), we easily obtain

$$\eta_0\eta_r = -e_{r,\, c} + (e_{r,\, 0} - e_{r,\, c})\eta_0 + (e_{r,\, 1} - e_{r,\, c})\eta_1 + \ldots.,$$

$$\eta_0\eta_c = -e_{c,\, r} + (e_{c,\, 0} - e_{c,\, r})\eta_0 + (e_{c,\, 1} - e_{c,\, r})\eta_1 + \ldots.,$$

$$\begin{aligned} &-e_{r,\, c}\eta_c + (e_{r,\, 0} - e_{r,\, c})\eta_0\eta_c + (e_{r,\, 1} - e_{r,\, c})\eta_1\eta_c + \ldots. \\ = &-e_{c,\, r}\eta_r + (e_{c,\, 0} - e_{c,\, r})\eta_0\eta_r + (e_{c,\, 1} - e_{c,\, r})\eta_1\eta_r + \ldots. \end{aligned}$$

Making $\chi = 1$ in this equation, which may be done provided neither r nor c is zero, we obtain

$$e_{c,\, r} = e_{r,\, c}. \tag{ix}$$

If, now, we write

$$\eta_0^2 = (e_{0,\, 0} - n - 1)\eta_0 + (e_{0,\, 1} - n)\eta_1 + (e_{0,\, 2} - n)\eta_2 + \ldots. + (e_{0,\, m-1} - n)\eta_{m-1},$$

$$e_{m+h} = e_h \text{ and } e_{-h} = e_{m-h},$$

equations (v), (viii) and (ix) become

$$\left.\begin{aligned} &e_{r,\, 0} + e_{r,\, 1} + e_{r,\, 2} + \ldots. + e_{r,\, m-1} = n, \\ &e_{r,\, c} = e_{c,\, r} = e_{m-r,\, c-r} = e_{c-r,\, m-r} = e_{m-c+r,\, m-c} = e_{m-c,\, m-c+r}, \\ &\qquad r = 0, 1, 2, 3, \ldots., m-1 \\ &\qquad c = 0, 1, 2, 3, \ldots., m-1 \end{aligned}\right\} \tag{x}$$

Returning to equations (iii) and (iv),

$$\begin{aligned} (\eta_0\eta_r)\eta_c &= -e_{r,\, c}\eta_c + (e_{r,\, 0} - e_{r,\, c})\eta_0\eta_c + (e_{r,\, 1} - e_{r,\, c})\eta_1\eta_c + \ldots. \\ = (\eta_r\eta_c)\eta_0 &= -e_{c-r,\, m-r}\eta_0 + (e_{c-r,\, m-r+1} - e_{c-r,\, m-r})\eta_0\eta_1 + \ldots. \end{aligned}$$

Substituting for the products $\eta_0\eta_1$, $\eta_0\eta_2$, ..., $\eta_0\eta_c$, $\eta_1\eta_2$, ..., their linear values as given in equation (iv)$(\eta_0\eta_1\eta_2 \ldots. \eta_{m-1})^h$, we obtain

$$\begin{aligned} &(e_{r,\, 0} - e_{r,\, c})e_{c,\, 0} + (e_{r,\, 1} - e_{r,\, c})e_{c-1,\, m-1} + (e_{r,\, 2} - e_{r,\, c})e_{c-2,\, m-2} + \ldots. \\ = &-e_{c-r,\, m-r} + (e_{c-r,\, m-r+1} - e_{c-r,\, m-r})e_{1,\, 0} + (e_{c-r,\, m-r+2} - e_{c-r,\, m-r})e_{2,\, 0} + \ldots. \\ \therefore\ &e_{r,\, c} + (e_{r,\, 0} - e_{r,\, c})e_{0,\, c} + (e_{r,\, 1} - e_{r,\, c})e_{1,\, c} + (e_{r,\, 2} - e_{r,\, c})e_{2,\, c} + \ldots. \\ = &(e_{r-1,\, c-1} - e_{r,\, c})e_{1,\, 0} + (e_{r-2,\, c-2} - e_{r,\, c})e_{2,\, 0} + (e_{r-3,\, c-3} - e_{r,\, c})e_{3,\, 0} + \ldots. \end{aligned} \tag{xi}$$

in which $c > r > 0$.

It may now be seen that if $m > 3$ and the equations $e_{r,\, c} = e_{c,\, r} = \ldots.$ of (x) be considered to be identities and therefore be not counted as equations proper, the number of e-coefficients will be $\frac{1}{6}(m+1)(m+2)$, the number of independent linear equations connecting these will be $\frac{1}{2}(m+1)$, and the number of independent quadratic equations connecting them will be $\frac{1}{6}(m-1)(m-2)$. The coefficient $e_{0,0}$ occurs only in the equation

$$e_{0,0}+e_{0,1}+e_{0,2}+\ldots\ldots+e_{0,m-1}=n.$$

Omitting this equation and the coefficient $e_{0,0}$, there will remain $\frac{1}{6}(m-1)(m+4)$ distinct e-coefficients connected by $\frac{1}{2}(m-1)$ linear and $\frac{1}{6}(m-1)(m-2)$ quadratic equations. These e-coefficients may most conveniently be considered as forming a rectangular array of elements in $m-1$ rows and m columns such that

$$\begin{aligned} e_{r,c} &= e_{c-r,\,m-r} = e_{m-c,\,m-c+r} \\ &= e_{c,r} = e_{m-r,\,c-r} = e_{m-c+r,\,m-c}, \end{aligned}$$

the rows numbering $1, 2, 3, \ldots\ldots, m-1$, the columns numbering $0, 1, 2, 3, \ldots\ldots, m-1$. In the following examples, this arrangement is adopted, but the letters a, b, c, d with single subscript indices are used for convenience instead of the single letter e with double subscript indices.

EXAMPLES.

$m=3$.

$$\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}(\eta_0)=\begin{pmatrix}a_1 & a_2 & b_1\\ a_2 & b_1 & a_1\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}.$$

$$\begin{array}{llll} \text{No. of row,} \quad . \quad . \quad . & (0) & 2 & 1 \\ \text{Subscript of } b, \quad . \quad . \quad . & - & 1 & 1 \end{array}\Big\}$$

$$S=\{(a_1a_2)(b_1)\},$$

$$a_1+a_2+b_1=n,$$

$$b_1+(a_1-b_1)(a_2-a_1)+(a_2-b_1)(b_1-a_2)=0.$$

One linear and one quadratic equation.

[Legendre, "Essai sur la Théorie des Nombres," 2e edition, §480; or, "Théorie des Nombres," 3e edition, §513. Cayley, Collected Mathematical Papers, vol. XI, pp. 86–89, or Proceedings of the London Mathematical Society, vol. XI (1880), pp. 7–9.]

$m = 5$.

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix}(\eta_0) = \begin{pmatrix} a_1 & a_4 & b_1 & b_2 & b_1\\ a_2 & b_1 & a_3 & b_2 & b_2\\ a_3 & b_2 & b_2 & a_2 & b_1\\ a_4 & b_1 & b_2 & b_1 & a_1\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix}.$$

No. of row, . . .	(0)	3	1	4	2
Subscript of b, . .		1	1	2	2

$$S = \{(a_1a_2a_4a_3)(b_1b_2)\},$$

$$(a_1 + a_4 + 2b_1 + b_2)\, S^h = n,$$

$$\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_4 - b_1)(b_1 - a_2) + (b_2 - b_1)(b_2 - a_4)\}\, S^h = 0.$$

Two linear and two quadratic equations.

[Legendre, "Théorie des Nombres," 3^e edition, §§523–527. Cayley, Collected Mathematical Papers, vol. XI, pp. 314 and 315, and vol. XII, pp. 72 and 73; or Proceedings of the London Mathematical Society, vol. XII (1881), pp. 15 and 16, and vol. XVI (1885), pp. 61–63. Carey (F. S.), Trinity Fellowship Dissertation, 1884. The present writer has not seen Mr. Carey's Dissertation itself, but the results arrived at by Mr. Carey were quoted by Professor Cayley in his paper of 1885.]

$m = 7$.

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\end{pmatrix}(\eta_0) = \begin{pmatrix} a_1 & a_6 & b_1 & c_1 & b_3 & c_2 & b_1\\ a_2 & b_1 & a_5 & c_2 & b_2 & b_2 & c_1\\ a_3 & c_1 & c_2 & a_4 & b_3 & b_2 & b_3\\ a_4 & b_3 & b_2 & b_3 & a_3 & c_1 & c_2\\ a_5 & c_2 & b_2 & b_2 & c_1 & a_2 & b_1\\ a_6 & b_1 & c_1 & b_3 & c_2 & b_1 & a_1\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\end{pmatrix}.$$

No. of row, . . .	(0)	4	1	5	2	6	3
Subscript of b, . .	—	3	1	2	2	1	3
No. of row, . ,	(0)	5	3	1	6	4	2
Subscript of c, . .	—	2	2	1	2	1	1

$$S = \{(a_1 a_3 a_2 a_6 a_4 a_5)(b_1 b_3 b_2)(c_1 c_2)\},$$

$$(a_1 + a_6 + 2b_1 + c_1 + b_3 + c_2)\, S^h = n,$$

$$\begin{aligned}\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_6 - b_1)(b_1 - a_2) + (c_1 - b_1)(c_2 - a_4) \\ + (b_3 - b_1)(b_2 - a_5) + (c_2 - b_1)(b_2 - a_6)\}\, S^h = 0,\end{aligned}$$

$$\begin{aligned}\{c_1 + (a_1 - c_1)\, a_3 + (a_6 - c)\, c_1 + (b_1 - c_1)\, c_2 + (b_3 - c_1)\, b_3 + (c_2 - c_1)\, b_2 \\ + (b_1 - c_1)\, b_3 - (a_2 - c_1)\, a_1 - (b_1 - c_1)\, a_2 - (a_5 - c_1)\, a_3 \\ - (c_2 - c_1)\, a_4 - (b_2 - c_1)\, a_5 - (b_2 - c_1)\, a_6\}\, S^h = 0.\end{aligned}$$

Three linear and five (= 3 + 2) quadratic equations.

TABLE I.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6
29	4	$\eta_0\eta_1$	1	0	1	0	0	1	1
		$\eta_0\eta_2$	0	1	0	1	1	1	0
		$\eta_0\eta_3$	0	0	1	2	0	1	0
43	6	$\eta_0\eta_1$	0	1	1	1	0	2	1
		$\eta_0\eta_2$	0	1	0	2	1	1	1
		$\eta_0\eta_3$	0	1	2	2	0	1	0
71	10	$\eta_0\eta_1$	2	2	2	0	1	1	2
		$\eta_0\eta_2$	1	2	0	1	3	3	0
		$\eta_0\eta_3$	2	0	1	2	1	3	1

$m = 11.$

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\end{pmatrix}(\eta_0) = \begin{pmatrix} a_1 & a_{10} & b_1 & c_1 & c_4 & c_6 & b_5 & c_5 & c_7 & c_{10} & b_1 \\ a_2 & b_1 & a_9 & c_{10} & b_2 & c_3 & c_2 & c_9 & c_8 & b_2 & c_1 \\ a_3 & c_1 & c_{10} & a_8 & c_7 & c_8 & b_3 & b_4 & b_3 & c_3 & c_4 \\ a_4 & c_4 & b_2 & c_7 & a_7 & c_5 & c_9 & b_4 & b_4 & c_2 & c_6 \\ a_5 & c_6 & c_3 & c_8 & c_5 & a_6 & b_5 & c_2 & b_3 & c_9 & b_5 \\ a_6 & b_5 & c_2 & b_3 & c_9 & b_5 & a_5 & c_6 & c_3 & c_8 & c_5 \\ a_7 & c_5 & c_9 & b_4 & b_4 & c_2 & c_6 & a_4 & c_4 & b_2 & c_7 \\ a_8 & c_7 & c_8 & b_3 & b_4 & b_3 & c_3 & c_4 & a_3 & c_1 & c_{10} \\ a_9 & c_{10} & c_2 & c_3 & c_2 & c_9 & c_8 & b_2 & c_1 & a_2 & b_1 \\ a_{10} & b_1 & c_1 & c_4 & c_6 & b_5 & c_5 & c_7 & c_{10} & b_1 & a_1 \end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\end{pmatrix}$$

No. of row, . .	(0)	6	1	7	2	8	3	9	4	10	5
Subscript of b, . .	—	5	1	4	2	3	3	2	4	1	5
No. of row, . Subscript of c, .	—	4	8	1	5	9	2	6	10	3	7

$$S = \{(a_1 a_2 a_4 a_8 a_5 a_{10} a_9 a_7 a_3 a_6)(b_1 b_2 b_4 b_3 b_5)(c_1 c_2 c_4 c_8 c_5 c_{10} c_9 c_7 c_3 c_6)\}$$

$$(a_1 + a_{10} + 2b_1 + c_1 + c_4 + c_6 + b_5 + c_5 + c_7 + c_{10})\, S^h = n,$$

$$\begin{aligned}\{b_1 + (a_1 - b_1)(a_2 - a_1) &+ (a_{10} - b_1)(b_1 - a_2) + (c_1 - b_1)(c_{10} - a_4) \\ &+ (c_4 - b_1)(b_2 - a_5) + (c_6 - b_1)(c_3 - a_6) + (b_5 - b_1)(c_2 - a_7) \\ &+ (c_5 - b_1)(c_9 - a_8) + (c_7 - b_1)(c_8 - a_9) \\ &+ (c_{10} - b_1)(b_2 - a_{10})\}\, S^h = 0,\end{aligned}$$

$$\begin{aligned}\{c_1 + (a_1 - c_1)\, a_3 &+ (a_{10} - c_1)\, c_1 + (b_1 - c_1)\, c_{10} + \dots \\ &- (a_2 - c_1)\, a_1 - (b_1 - c_1)\, a_2 - (a_9 - c_1)\, a_3 - \dots\}\, S^h = 0.\end{aligned}$$

Five linear and fifteen (= 5 + 10) quadratic equations.

TABLE II.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}
23	2	$\eta_1\eta_0$	1	0	0	0	0	0	0	0	1	0	0
		$\eta_0\eta_2$	0	0	0	0	0	0	1	0	1	0	0
		$\eta_0\eta_3$	0	0	0	0	1	1	0	0	0	0	0
		$\eta_0\eta_4$	0	0	0	1	0	0	0	0	0	1	0
		$\eta_0\eta_5$	0	0	0	1	0	0	0	1	0	0	0
67	6	$\eta_0\eta_1$	1	0	0	2	0	1	1	1	0	0	0
			0	0	0	0	1	0	1	0	1	1	2
			2	2	0	0	0	1	0	1	0	0	0
			0	0	1	0	0	1	0	1	1	1	1
			0	1	0	1	1	0	1	1	0	0	1
89	8	$\eta_0\eta_1$	0	2	1	1	1	0	0	2	0	0	1
			0	1	0	0	1	1	2	0	1	1	1
			2	1	0	0	0	1	1	0	1	1	1
			2	1	1	0	0	2	0	0	0	2	0
			0	0	1	1	2	1	0	2	1	0	0

$m = 13.$

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\\ \eta_{11}\\ \eta_{12}\end{pmatrix}(\eta_0) = \begin{pmatrix}
a_1 & a_{12} & b_1 & c_1 & d_1 & c_4 & c_7 & b_6 & c_6 & c_9 & d_4 & c_{12} & b_1\\
a_2 & b_1 & a_{11} & c_{12} & b_2 & c_5 & c_2 & d_3 & d_2 & c_{11} & c_8 & b_2 & c_1\\
a_3 & c_1 & c_{12} & a_{10} & d_4 & c_8 & b_3 & c_{10} & b_5 & c_3 & b_3 & c_5 & d_1\\
a_4 & d_1 & b_2 & d_4 & a_9 & c_9 & c_{11} & c_3 & b_4 & b_4 & c_{10} & c_2 & c_4\\
a_5 & c_4 & c_5 & c_8 & c_9 & a_8 & c_6 & d_2 & b_5 & b_4 & b_5 & d_3 & c_7\\
a_6 & c_7 & c_2 & b_3 & c_{11} & c_6 & a_7 & b_6 & d_3 & c_{10} & c_3 & d_2 & b_6\\
a_7 & b_6 & d_3 & c_{10} & c_3 & d_2 & b_6 & a_6 & c_7 & c_2 & b_3 & c_{11} & c_6\\
a_8 & c_6 & d_2 & b_5 & b_4 & b_5 & d_3 & c_7 & a_5 & c_4 & c_5 & c_8 & c_9\\
a_9 & c_9 & c_{11} & c_3 & b_4 & b_4 & c_{10} & c_2 & c_4 & a_4 & d_1 & b_2 & d_4\\
a_{10} & d_4 & c_8 & b_3 & c_{10} & b_5 & c_3 & b_3 & c_5 & d_1 & a_3 & c_1 & c_{12}\\
a_{11} & c_{12} & b_2 & c_5 & c_2 & d_3 & d_2 & c_{11} & c_8 & b_2 & c_1 & a_2 & b_1\\
a_{12} & b_1 & c_1 & d_1 & c_4 & c_7 & b_6 & c_6 & c_9 & d_4 & c_{12} & b_1 & a_1
\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\\ \eta_{11}\\ \eta_{12}\end{pmatrix}$$

No. of row, . .	(0)	7	1	8	2	9	3	10	4	11	5	12	6
Subscript of b, . .	—	6	1	5	2	4	3	3	4	2	5	1	6

No. of row, . . Subscript of c, . .	—	9	5	1	10	6	2	11	7	3	12	8	4

No. of row, . .	(0)	10	7	4	1	11	8	5	2	12	9	6	3
Subscript of d, . .	—	4	3	4	1	3	3	2	2	4	1	2	1

$$S = \{(a_1a_2a_4a_8a_3a_6a_{12}a_{11}a_9a_5a_{10}a_7)(b_1b_2b_4b_5b_3b_6)(c_1c_2c_4c_8c_3c_6c_{12}c_{11}c_9c_5c_{10}c_7)(d_1d_2d_4d_3)\},$$

$$(a_1 + a_{12} + 2b_1 + c_1 + d_1 + c_4 + c_7 + b_6 + c_6 + c_9 + d_4 + c_{12})\, S^h = n.$$

$$\begin{aligned}&\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_{12} - b_1)(b_1 - a_2) + (c_1 - b_1)(c_{12} - a_3)\\ &\qquad\qquad + (d_1 - b_1)(b_2 - a_4) + \ldots\}\, S^h = 0,\end{aligned}$$

$$\begin{aligned}&\{c_1 + (a_1 - c_1)\,a_3 + (a_{12} - c_1)\,c_1 + (b_1 - c_1)\,c_{12} + (d_1 - c_1)\,d_4 + \ldots\\ &\qquad - (a_2 - c_1)\,a_1 - (b_1 - c_1)\,a_2 - (a_{11} - c_1)\,a_3 - (c_{12} - c_1)\,a_4 - \ldots\}\, S^h = 0,\end{aligned}$$

$$\begin{aligned}&\{d_1 + (a_1 - d_1)\,a_4 + (a_{12} - d_1)\,d_1 + (b_1 - d_1)\,b_2 + (c_1 - d_1)\,d_4 + \ldots\\ &\qquad - (a_3 - d_1)\,a_1 - (c_1 - d_1)\,a_2 - (c_{12} - d_1)\,a_3 - (a_{10} - d_1)\,a_4 - \ldots\}\, S^h = 0.\end{aligned}$$

Six linear and twenty-two (= 6 + 12 + 4) quadratic equations.

TABLE III.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}	η_{11}	η_{12}
53	4	$\eta_0\eta_1$	1	0	0	1	1	1	0	0	0	0	0	0	0
		$\eta_0\eta_2$	0	0	0	0	1	0	0	0	1	0	0	1	1
		$\eta_0\eta_3$	0	1	0	0	0	0	0	1	1	0	0	0	1
		$\eta_0\eta_4$	0	1	1	0	0	0	0	0	0	0	1	0	1
		$\eta_0\eta_5$	0	1	0	0	0	0	0	1	1	0	1	0	0
		$\eta_0\eta_6$	0	0	0	0	0	0	2	0	0	1	0	1	0
79	6	$\eta_0\eta_1$	0	0	1	1	1	1	0	0	1	0	0	0	1
			0	1	0	0	1	1	0	1	0	0	0	1	1
			0	1	0	0	0	0	0	0	1	2	0	1	1
			1	1	1	0	0	0	0	2	0	0	0	0	1
			0	1	1	0	0	0	1	0	1	0	1	1	0
			0	0	0	0	0	1	2	0	1	0	2	0	0
131	10	$\eta_0\eta_1$	0	0	1	2	2	1	0	1	0	2	0	0	1
			2	1	0	0	0	1	0	1	1	0	2	0	2
			0	2	0	0	0	2	1	0	0	1	1	1	2
			0	2	0	0	2	2	0	1	1	1	0	0	1
			1	1	1	2	2	0	0	1	0	1	0	1	0
			2	0	0	1	0	0	2	1	1	0	1	1	1

$m = 17.$

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\\ \eta_{11}\\ \eta_{12}\\ \eta_{13}\\ \eta_{14}\\ \eta_{15}\\ \eta_{16}\end{pmatrix}(\eta_0) = \begin{pmatrix}
a_1 & a_{16} & b_1 & c_1 & d_1 & d_4 & c_6 & d_{11} & c_9 & b_8 & c_8 & d_6 & c_{11} & d_{13} & d_{16} & c_{16} & b_1\\
a_2 & b_1 & a_{15} & c_{16} & b_2 & d_{12} & c_2 & c_5 & d_2 & d_9 & d_8 & d_{15} & c_{12} & c_{15} & d_5 & b_2 & c_1\\
a_3 & c_1 & c_{16} & a_{14} & d_{16} & d_5 & b_3 & c_{10} & d_{14} & c_3 & b_7 & c_{14} & d_3 & c_7 & b_3 & d_{12} & d_1\\
a_4 & d_1 & b_2 & d_{16} & a_{13} & d_{13} & c_{15} & c_7 & b_4 & c_{13} & d_7 & d_{10} & c_4 & b_4 & c_{10} & c_2 & d_4\\
a_5 & d_4 & d_{12} & d_5 & d_{13} & a_{12} & c_{11} & c_{12} & d_3 & c_4 & b_5 & b_6 & b_5 & c_{13} & d_{14} & c_5 & c_6\\
a_6 & c_6 & c_2 & b_3 & c_{15} & c_{11} & a_{11} & d_6 & d_{15} & c_{14} & d_{10} & b_6 & b_6 & d_7 & c_3 & d_2 & d_{11}\\
a_7 & d_{11} & c_5 & c_{10} & c_7 & c_{12} & d_6 & a_{10} & c_8 & d_8 & b_7 & d_7 & b_5 & d_{10} & b_7 & d_9 & c_9\\
a_8 & c_9 & d_2 & d_{14} & b_4 & d_3 & d_{15} & c_8 & a_9 & b_8 & d_9 & c_3 & c_{13} & c_4 & c_{14} & d_8 & b_8\\
a_9 & b_8 & d_9 & c_3 & c_{13} & c_4 & c_{14} & d_8 & b_8 & a_8 & c_9 & d_2 & d_{14} & b_4 & d_3 & d_{15} & c_8\\
a_{10} & c_8 & d_8 & b_7 & d_7 & b_5 & d_{10} & b_7 & d_9 & c_9 & a_7 & d_{11} & c_5 & c_{10} & c_7 & c_{12} & d_6\\
a_{11} & d_6 & d_{15} & c_{14} & d_{10} & b_6 & b_6 & d_7 & c_3 & d_2 & d_{11} & a_6 & c_6 & c_2 & b_3 & c_{15} & c_{11}\\
a_{12} & c_{11} & c_{12} & d_3 & c_4 & b_5 & b_6 & b_5 & c_{13} & d_{14} & c_5 & c_6 & a_5 & d_4 & d_{12} & d_5 & d_{13}\\
a_{13} & d_{13} & c_{15} & c_7 & b_4 & c_{13} & d_7 & d_{10} & c_4 & b_4 & c_{10} & c_2 & d_4 & a_4 & d_1 & b_2 & d_{16}\\
a_{14} & d_{16} & d_5 & b_3 & c_{10} & d_{14} & c_3 & b_7 & c_{14} & d_3 & c_7 & b_3 & d_{12} & d_1 & a_3 & c_1 & c_{16}\\
a_{15} & c_{16} & b_2 & d_{12} & c_2 & c_5 & d_2 & d_9 & d_8 & d_{15} & c_{12} & c_{15} & d_5 & b_2 & c_1 & a_2 & b_1\\
a_{16} & b_1 & c_1 & d_1 & d_4 & c_6 & d_{11} & c_9 & b_8 & c_8 & d_6 & c_{11} & d_{13} & d_{16} & c_{16} & b_1 & a_1
\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\\ \eta_{11}\\ \eta_{12}\\ \eta_{13}\\ \eta_{14}\\ \eta_{15}\\ \eta_{16}\end{pmatrix}$$

No. of row, .	(0)	9	1	10	2	11	3	12	4	13	5	14	6	15	7	16	8
Subscript of b, .	—	8	1	7	2	6	3	5	4	4	5	3	6	2	7	1	8
No. of row, . Subscript of c, .	—	6	12	1	7	13	2	8	14	3	9	15	4	10	16	5	11
No. of row, . Subscript of d, .	—	13	9	5	1	14	10	6	2	15	11	7	3	16	12	8	4

$$S = \{(a_1a_3a_9a_{10}a_{13}a_5a_{15}a_{11}a_{16}a_{14}a_8a_7a_4a_{12}a_2a_6)(b_1b_3b_8b_7b_4b_5b_2b_6)$$
$$(c_1c_3c_9c_{10}c_{13}c_5c_{15}c_{11}c_{16}c_{14}c_8c_7c_4c_{12}c_2c_6)(d_1d_3d_9d_{10}d_{13}d_5d_{15}d_{11}d_{16}d_{14}d_8d_7d_4d_{12}d_2d_6)\}.$$

$$(a_1 + a_{16} + 2b_1 + c_1 + d_1 + d_4 + c_6 + d_{11} + c_9 + b_8 + c_8 + d_6 + c_{11}$$
$$+ d_{13} + d_{16} + c_{16})\, S^h = n,$$

$$\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_{16} - b_1)(b_1 - a_2)$$
$$+ (c_1 - b_1)(c_{16} - a_3) + (d_1 - b_1)(b_2 - a_4) + \ldots.\}\, S^h = 0,$$

$$\{c_1 + (a_1 - c_1)\, a_3 + (a_{16} - c_1)\, c_1 + (b_1 - c_1)\, c_{16} + (d_1 - c_1)\, d_{16} + \ldots.$$
$$- (a_2 - c_1)\, a_1 - (b_1 - c_1)\, a_2 - (a_{15} - c_1)\, a_3 - (c_{16} - c_1)\, a_4 - \ldots.\}\, S^h = 0,$$

$$\{d_1 + (a_1 - d_1)\, a_4 + (a_{16} - d_1)\, d_1 + (b_1 - d_1)\, b_2 + (c_1 - d_1)\, d_{16} + \ldots.$$
$$- (a_3 - d_1)\, a_1 - (c_1 - d_1)\, a_2 + (c_{16} - d_1)\, a_3 + (a_{14} - d_1)\, a_4 - \ldots.\}\, S^h = 0.$$

Eight linear and forty $(= 8 + 16 + 16)$ quadratic equations.

TABLE IV.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}	η_{11}	η_{12}	η_{13}	η_{14}	η_{15}	η_{16}
103	6	$\eta_0\eta_1$	1	0	0	1	0	0	0	0	0	1	0	1	0	2	0	0	0
		$\eta_0\eta_2$	0	0	0	0	0	0	1	1	1	1	0	0	1	0	0	0	1
		$\eta_0\eta_3$	0	1	0	0	0	0	1	0	0	1	0	1	0	1	1	0	0
		$\eta_0\eta_4$	0	0	0	0	2	2	0	1	0	0	0	0	0	0	0	1	0
		$\eta_0\eta_5$	0	0	0	0	2	0	0	1	0	0	1	0	1	0	0	1	0
		$\eta_0\eta_6$	0	0	1	1	0	0	0	1	0	1	0	0	0	0	1	1	0
		$\eta_0\eta_7$	0	0	1	0	1	1	1	0	0	0	0	0	1	0	0	1	0
		$\eta_0\eta_8$	0	0	1	0	0	0	0	0	0	1	1	1	0	0	1	0	1
137	8	$\eta_0\eta_1$	0	0	1	1	1	1	0	0	0	0	1	1	1	0	0	0	1
			0	1	0	0	0	0	1	1	0	0	1	0	1	2	0	0	1
			0	1	0	2	0	0	0	1	0	1	1	1	0	0	0	0	1
			0	1	0	0	0	0	2	0	0	2	0	0	0	0	1	1	1
			2	1	0	0	0	0	1	1	0	0	0	0	0	2	0	1	0
			0	0	1	0	2	1	1	1	0	1	0	0	0	0	1	0	0
			0	0	1	1	0	1	1	0	1	1	1	0	0	0	1	0	0
			0	0	0	0	0	0	0	1	2	0	0	1	2	0	1	1	0
239	14	$\eta_0\eta_1$	3	0	2	1	0	1	2	0	0	0	0	0	1	1	1	0	2
			0	2	2	0	1	0	2	1	0	2	1	0	0	0	1	1	1
			0	1	0	0	1	1	1	1	1	0	1	2	2	2	1	0	0
			0	0	1	1	0	1	0	2	1	1	1	1	0	1	1	2	1
			0	1	0	1	1	0	1	0	2	0	1	1	1	1	1	1	2
			0	2	2	1	0	1	2	0	0	2	1	1	1	1	0	0	0
			0	0	1	1	2	0	0	2	0	1	1	1	1	1	1	2	0
			0	0	0	1	1	2	0	0	4	0	2	0	1	0	2	1	0